Perfect Shuffle

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Diaconis and others have noted that a “random” shuffle may not be random. In fact, they have suggested that if one shuffles a 52 card poker deck “perfectly” eight times, the deck is returned to its original order.

We start by defining a deck of cards as

```
(new.deck <- 1:52)
```

```
##  [1]  1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18 19 20 21 22 23
## [24] 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46
## [47] 47 48 49 50 51 52
```

Create a Perfect Shuffle

Let us start by creating a function that shuffles a deck perfectly.

```
shuffle <- function(deck, perfect = FALSE){
  n <- length(deck)
  if (perfect){
    deck.shuffled <- rep(0, n)
    deck.shuffled[2*(1:(n/2)) - 1] <- deck[1:(n/2)] # righthand (top) cards
    deck.shuffled[2*(1:(n/2))] <- deck[(n/2 + 1):n] # lefthand (bottom) cards
    return(deck.shuffled)
  }else{
    return(deck[sample(1:n)])
  }
}
```

We now check to see that it shuffles correctly.

```
deck <- new.deck
print("Check an imperfect shuffle.")
```

```
## [1] "Check an imperfect shuffle."
```

```
shuffle(deck, FALSE)
```

```
##  [1] 24 40  5 39 42 17  2 13 44  9  8 50  4 34 45 26 47  1 12 46 51 48  6
## [24]  7 14 19 21 15 23 35 37 10 38 52 36 11 22 18 20 30  3 49 41 29 25 31
## [47] 43 28 33 16 32 27
```

```
print("Check a perfect shuffle.")
```

```
## [1] "Check a perfect shuffle."
```

```
shuffle(deck, TRUE)
```

```
##  [1]  1 27  2 28  3 29  4 30  5 31  6 32  7 33  8 34  9 35 10 36 11 37 12
## [24] 38 13 39 14 40 15 41 16 42 17 43 18 44 19 45 20 46 21 47 22 48 23 49
## [47] 24 50 25 51 26 52
```

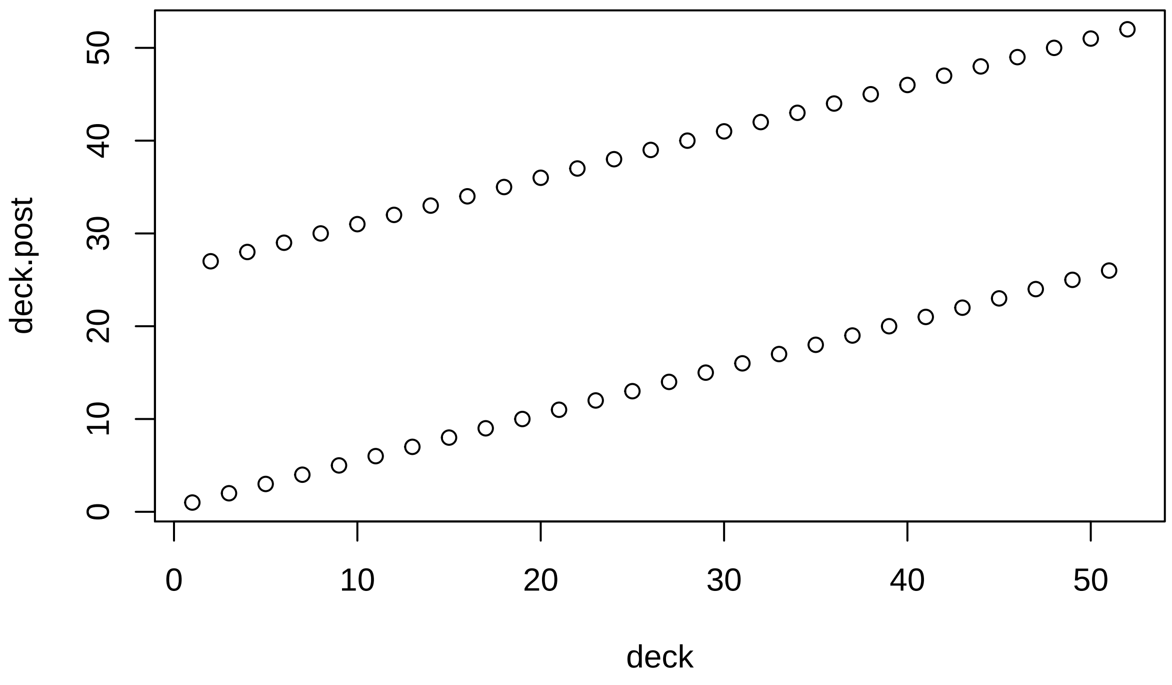
Next, we iterate through eight shuffles to check that we do get back to the original order.

```
deck.pre <- deck
print(deck)
```

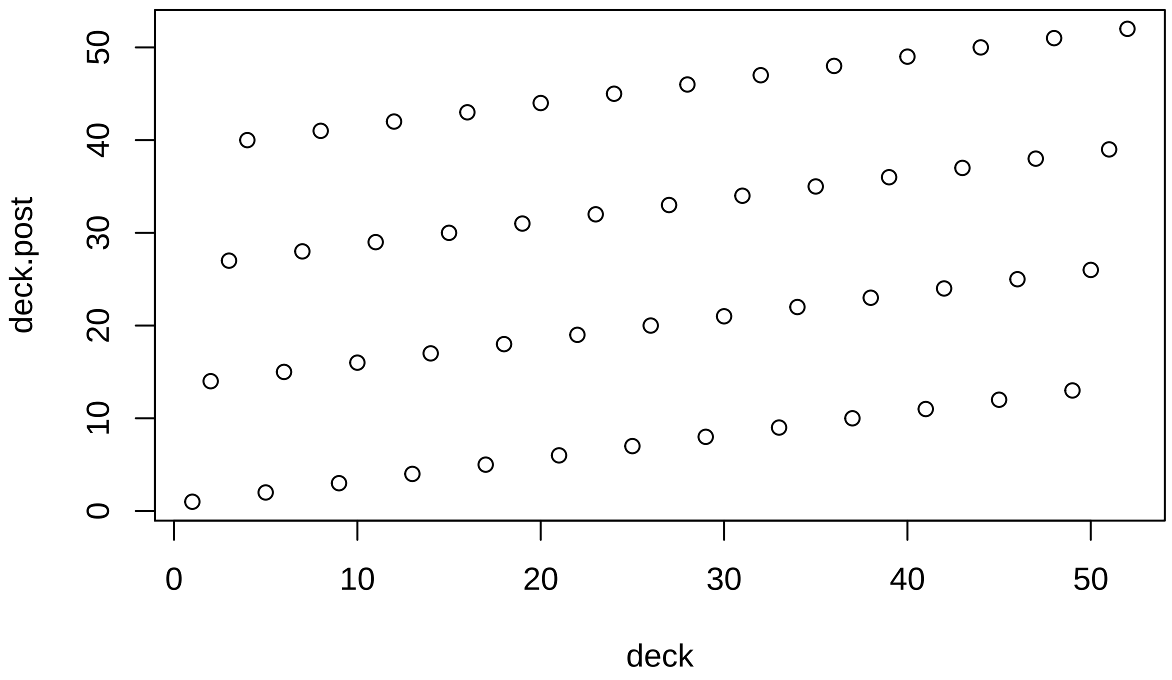
```
##  [1]  1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18 19 20 21 22 23
## [24] 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46
## [47] 47 48 49 50 51 52
```

```
for (i in 1:8){
  deck.post <- shuffle(deck.pre, perfect = TRUE)
  print(paste("Shuffles:", i))
  print(deck.post)
  plot(deck, deck.post)
  deck.pre <- deck.post
}
```

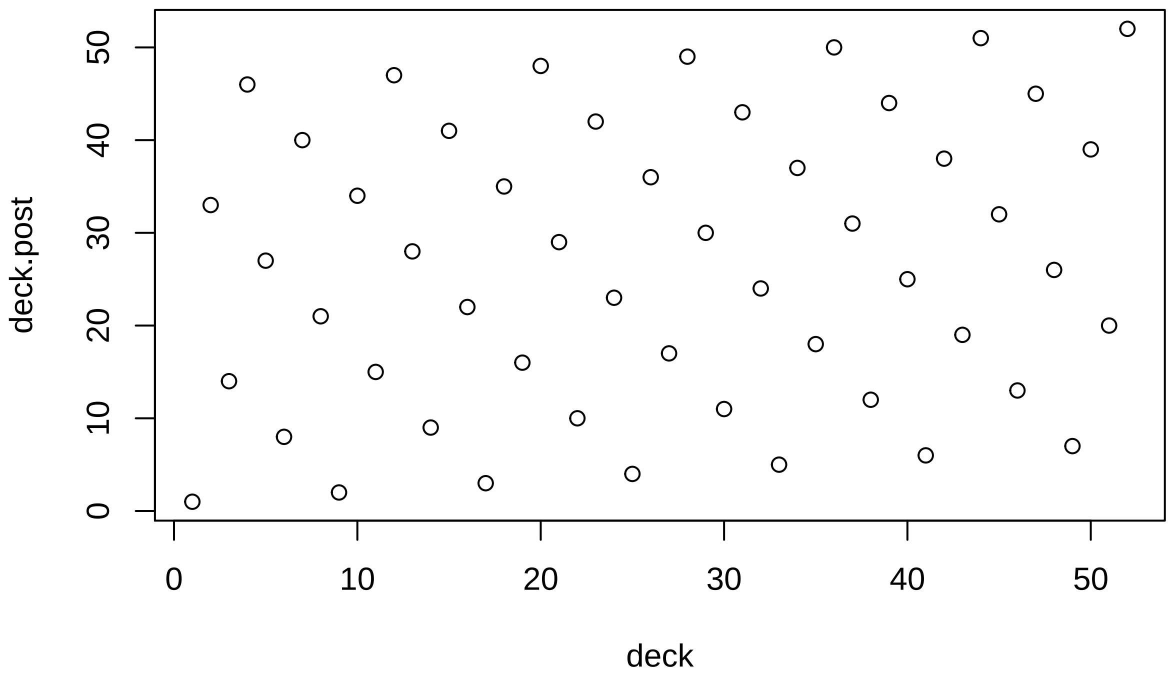
```
## [1] "Shuffles: 1"
##  [1]  1 27  2 28  3 29  4 30  5 31  6 32  7 33  8 34  9 35 10 36 11 37 12
## [24] 38 13 39 14 40 15 41 16 42 17 43 18 44 19 45 20 46 21 47 22 48 23 49
## [47] 24 50 25 51 26 52
```



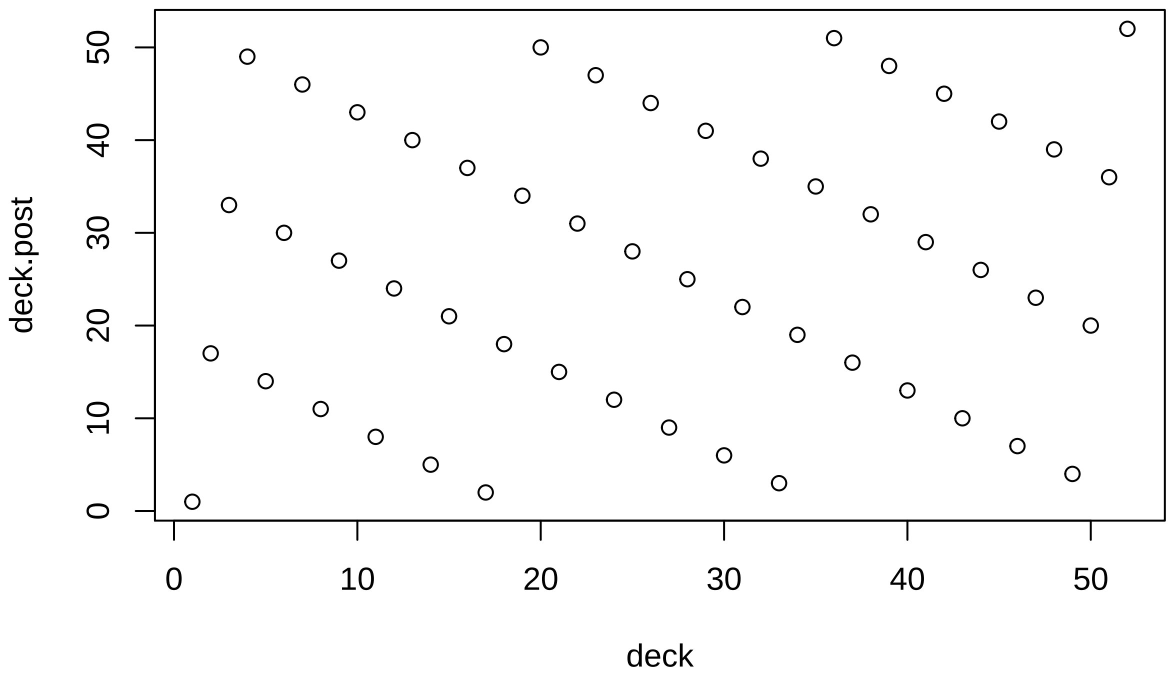
```
## [1] "Shuffles: 2"
##  [1]  1 14 27 40  2 15 28 41  3 16 29 42  4 17 30 43  5 18 31 44  6 19 32
## [24] 45  7 20 33 46  8 21 34 47  9 22 35 48 10 23 36 49 11 24 37 50 12 25
## [47] 38 51 13 26 39 52
```



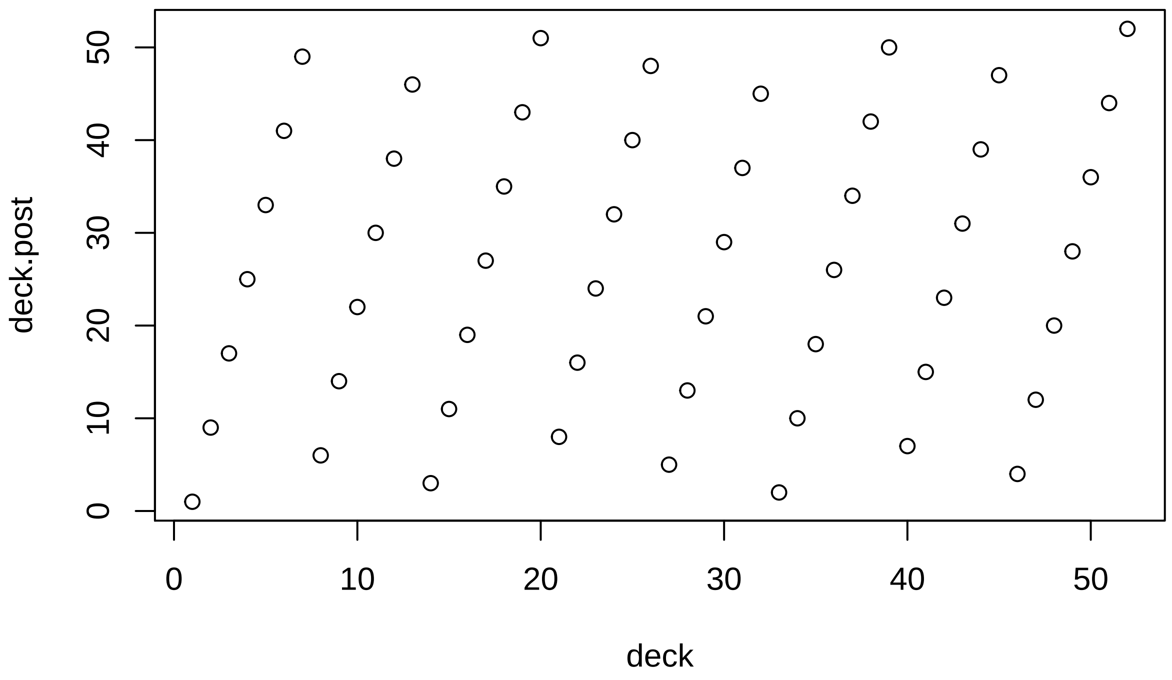
```
## [1] "Shuffles: 3"
##  [1]  1 33 14 46 27  8 40 21  2 34 15 47 28  9 41 22  3 35 16 48 29 10 42
## [24] 23  4 36 17 49 30 11 43 24  5 37 18 50 31 12 44 25  6 38 19 51 32 13
## [47] 45 26  7 39 20 52
```



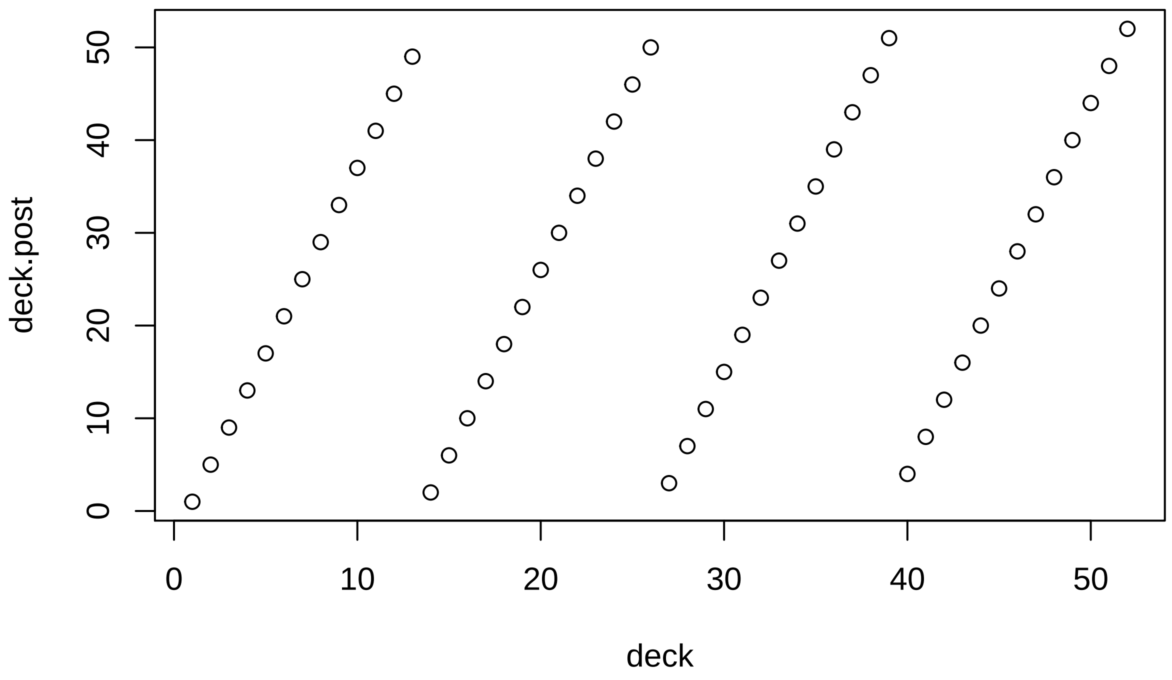
```
## [1] "Shuffles: 4"
##  [1]  1 17 33 49 14 30 46 11 27 43  8 24 40  5 21 37  2 18 34 50 15 31 47
## [24] 12 28 44  9 25 41  6 22 38  3 19 35 51 16 32 48 13 29 45 10 26 42  7
## [47] 23 39  4 20 36 52
```



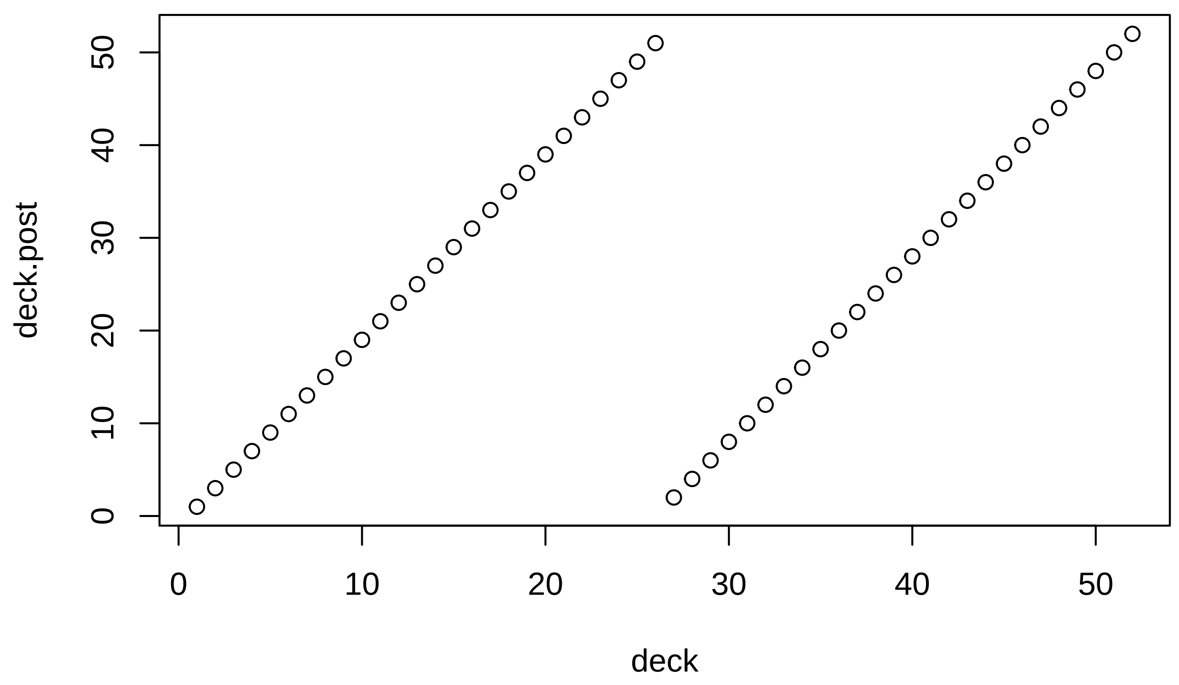
```
## [1] "Shuffles: 5"
##  [1]  1  9 17 25 33 41 49  6 14 22 30 38 46  3 11 19 27 35 43 51  8 16 24
## [24] 32 40 48  5 13 21 29 37 45  2 10 18 26 34 42 50  7 15 23 31 39 47  4
## [47] 12 20 28 36 44 52
```



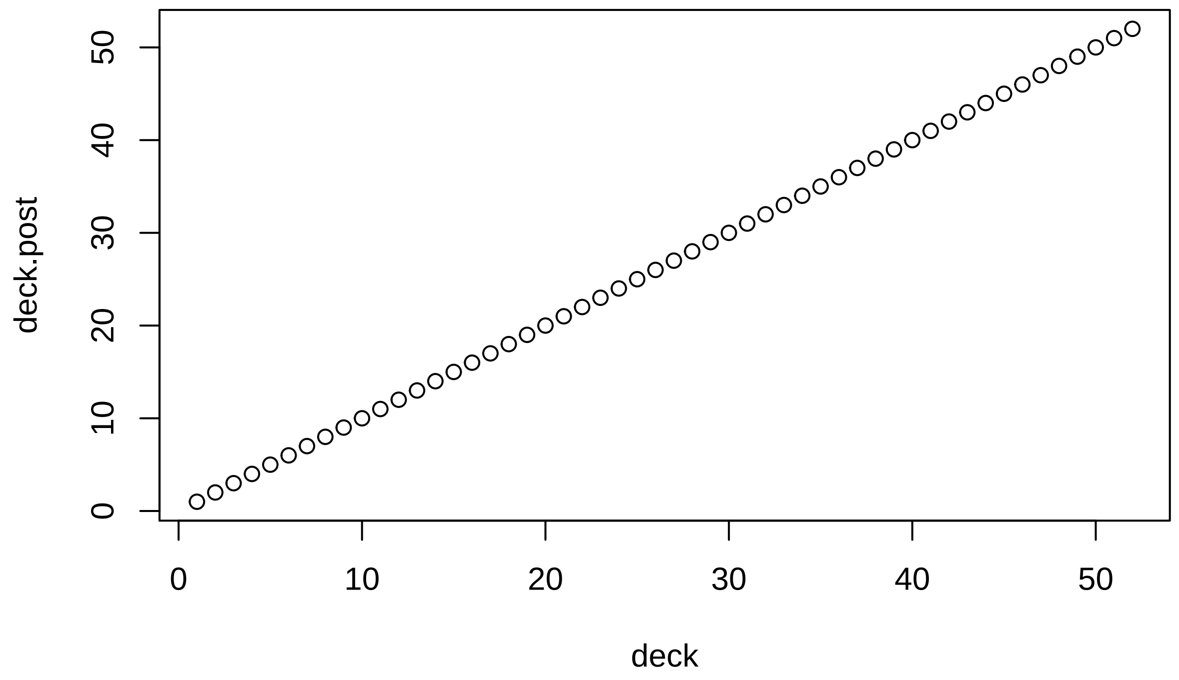
```
## [1] "Shuffles: 6"
##  [1]  1  5  9 13 17 21 25 29 33 37 41 45 49  2  6 10 14 18 22 26 30 34 38
## [24] 42 46 50  3  7 11 15 19 23 27 31 35 39 43 47 51  4  8 12 16 20 24 28
## [47] 32 36 40 44 48 52
```



```
## [1] "Shuffles: 7"
##  [1]  1  3  5  7  9 11 13 15 17 19 21 23 25 27 29 31 33 35 37 39 41 43 45
## [24] 47 49 51  2  4  6  8 10 12 14 16 18 20 22 24 26 28 30 32 34 36 38 40
## [47] 42 44 46 48 50 52
```



```
## [1] "Shuffles: 8"
##  [1]  1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18 19 20 21 22 23
## [24] 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46
## [47] 47 48 49 50 51 52
```



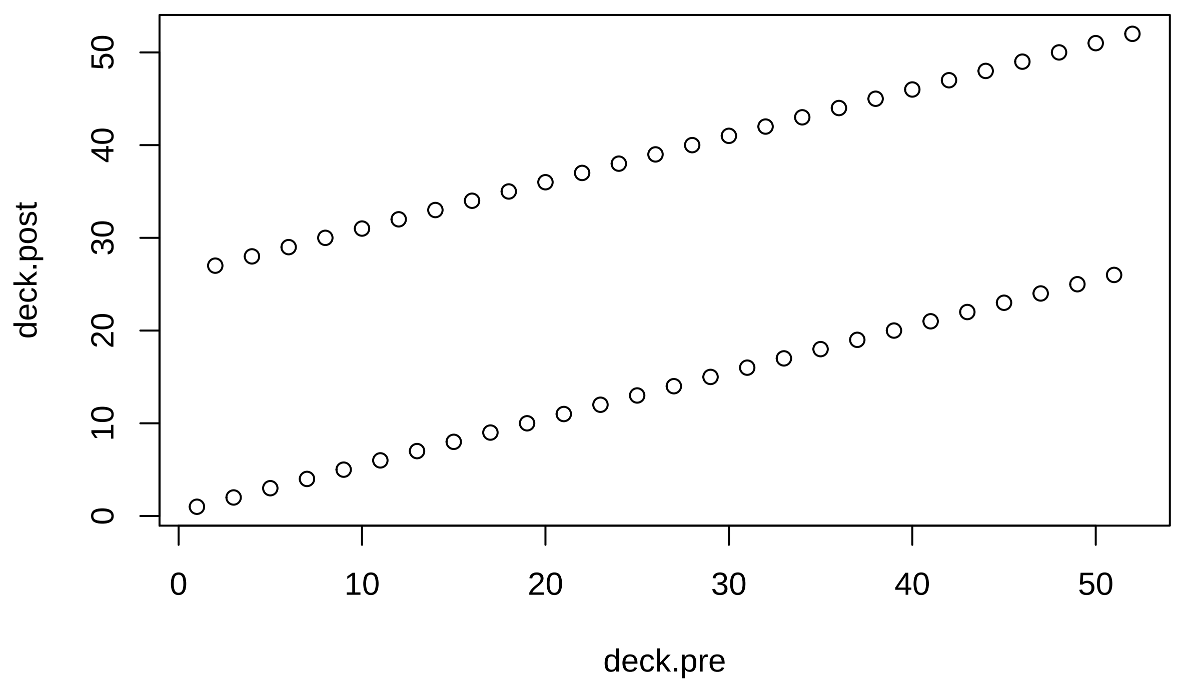
We repeat the above looking at the relationship of the deck to its previous state.

```r
deck.pre <- deck
print(deck)
```

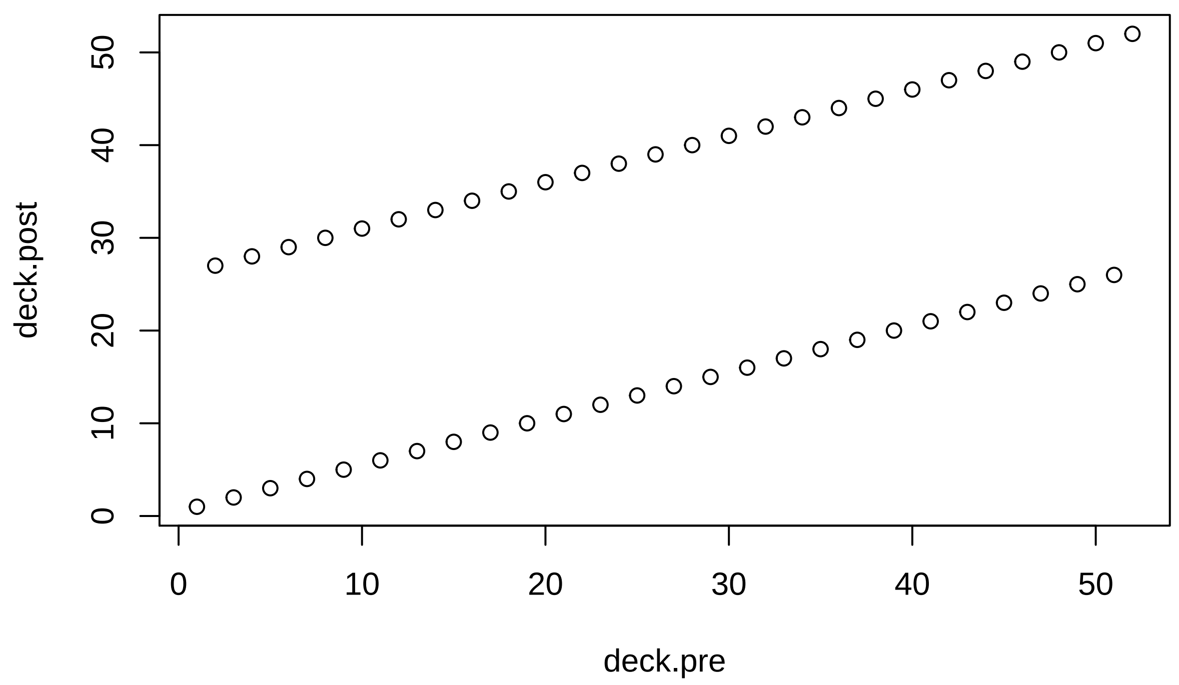
```
##  [1]  1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18 19 20 21 22 23
## [24] 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46
## [47] 47 48 49 50 51 52
```

```r
for (i in 1:8){
  deck.post <- shuffle(deck.pre, perfect = TRUE)
  print(paste("Shuffles:", i))
  print(deck.post)
  plot(deck.pre, deck.post)
  deck.pre <- deck.post
}
```

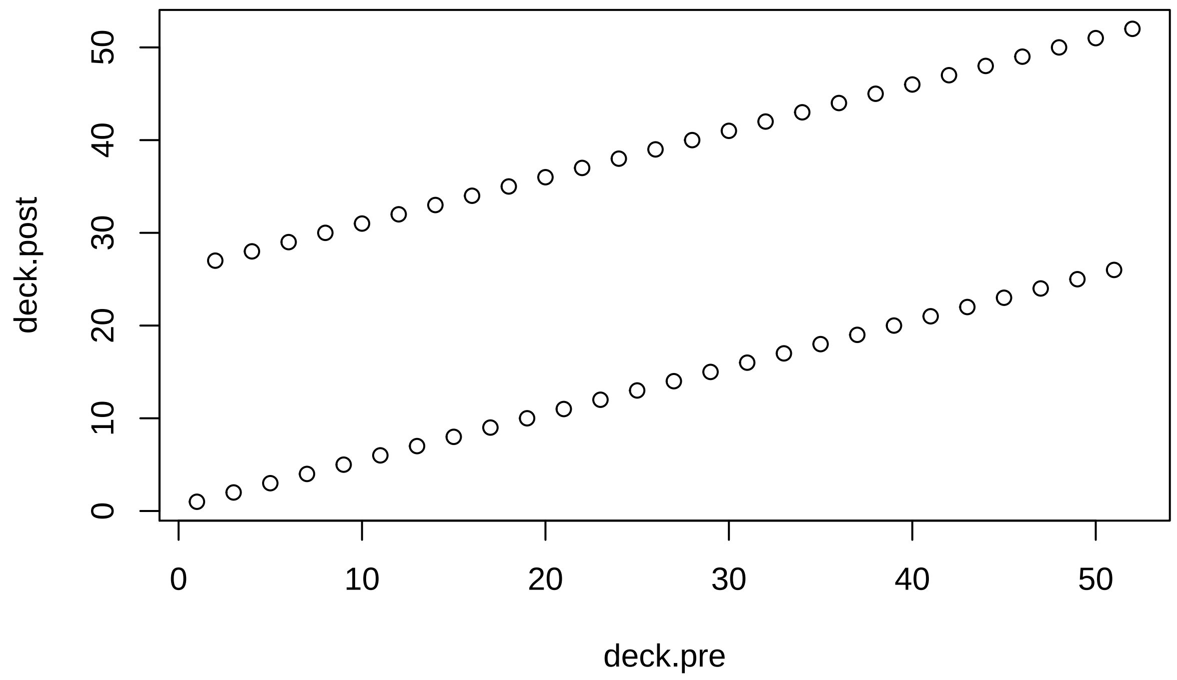
```
## [1] "Shuffles: 1"
##  [1]  1 27  2 28  3 29  4 30  5 31  6 32  7 33  8 34  9 35 10 36 11 37 12
## [24] 38 13 39 14 40 15 41 16 42 17 43 18 44 19 45 20 46 21 47 22 48 23 49
## [47] 24 50 25 51 26 52
```



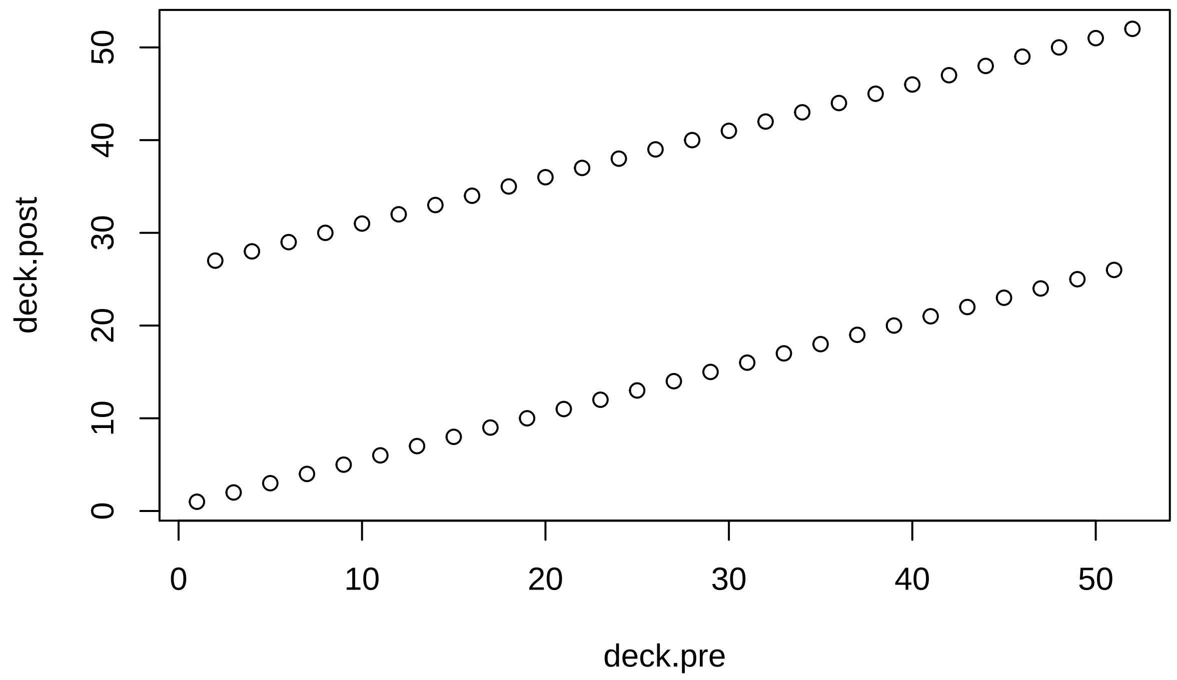
```
## [1] "Shuffles: 2"
##  [1]  1 14 27 40  2 15 28 41  3 16 29 42  4 17 30 43  5 18 31 44  6 19 32
## [24] 45  7 20 33 46  8 21 34 47  9 22 35 48 10 23 36 49 11 24 37 50 12 25
## [47] 38 51 13 26 39 52
```



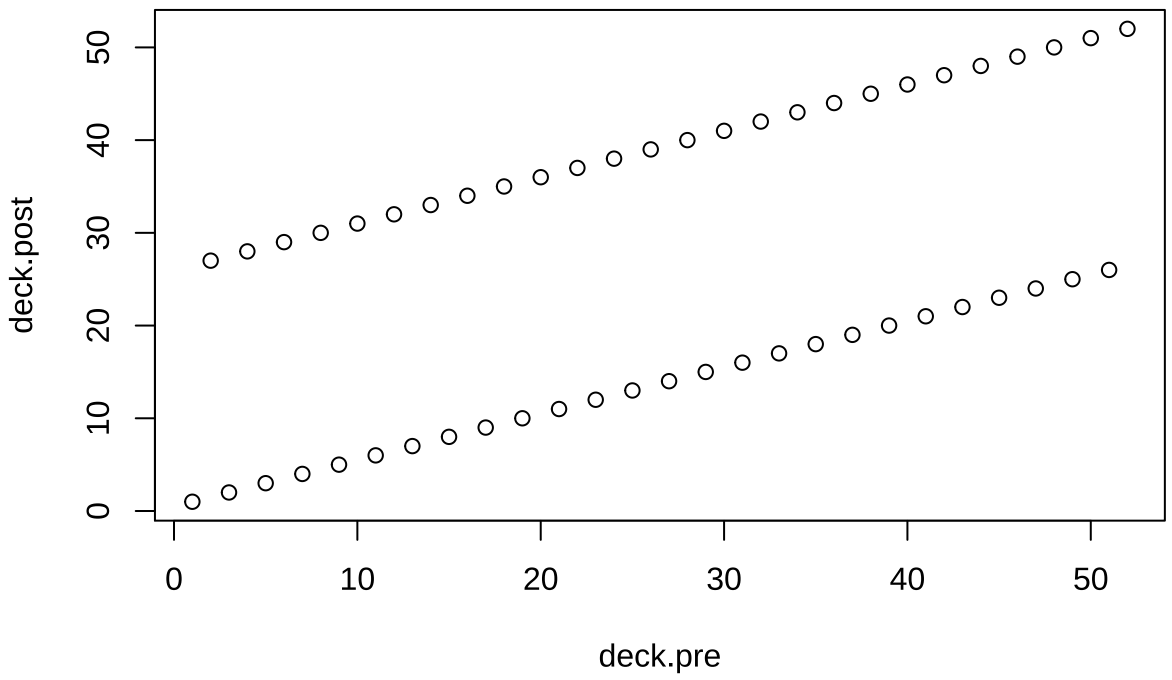
```
## [1] "Shuffles: 3"
##  [1]  1 33 14 46 27  8 40 21  2 34 15 47 28  9 41 22  3 35 16 48 29 10 42
## [24] 23  4 36 17 49 30 11 43 24  5 37 18 50 31 12 44 25  6 38 19 51 32 13
## [47] 45 26  7 39 20 52
```



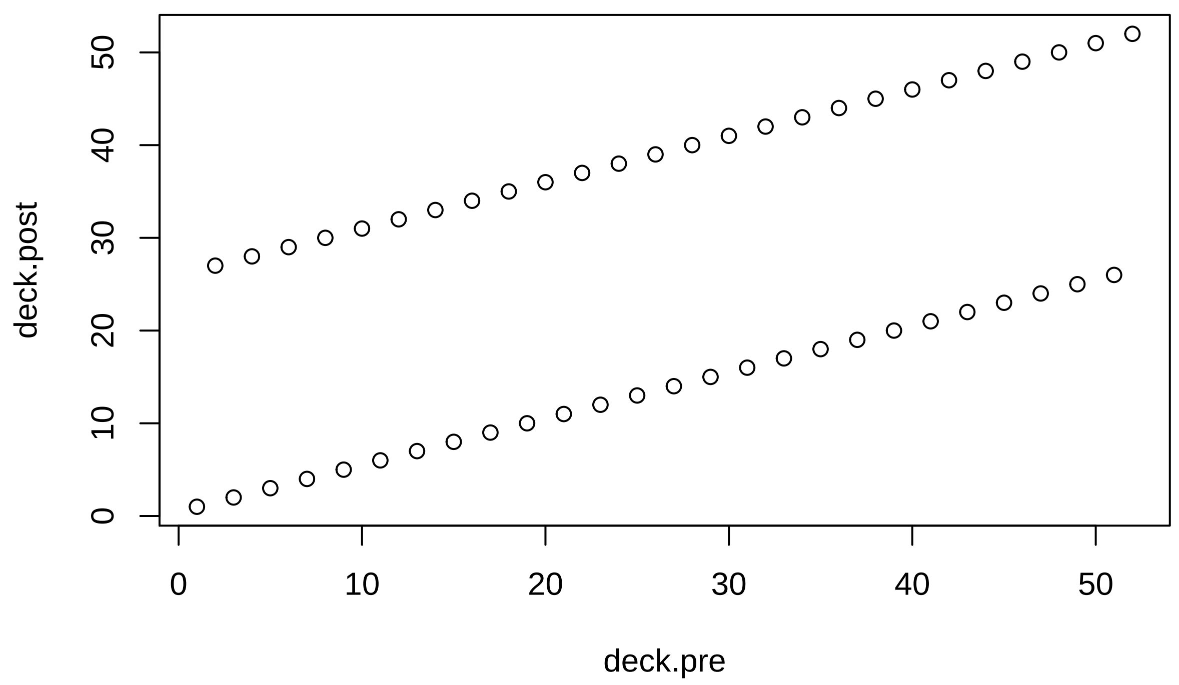
```
## [1] "Shuffles: 4"
##  [1]  1 17 33 49 14 30 46 11 27 43  8 24 40  5 21 37  2 18 34 50 15 31 47
## [24] 12 28 44  9 25 41  6 22 38  3 19 35 51 16 32 48 13 29 45 10 26 42  7
## [47] 23 39  4 20 36 52
```



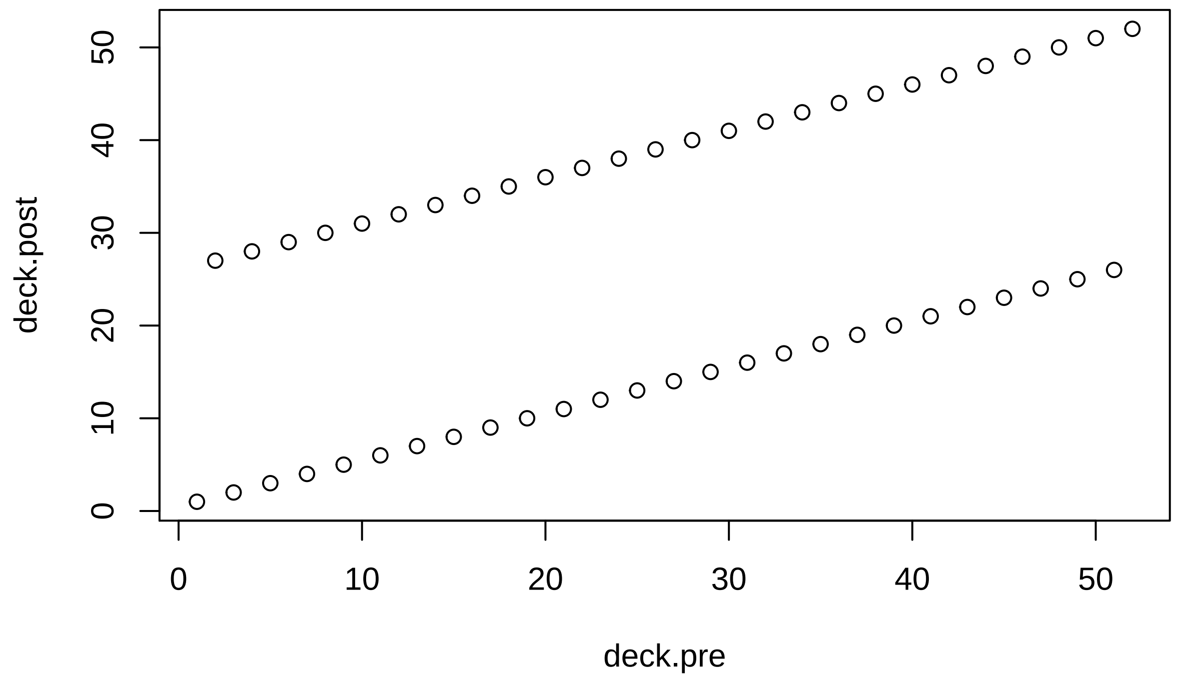
```
## [1] "Shuffles: 5"
##  [1]  1  9 17 25 33 41 49  6 14 22 30 38 46  3 11 19 27 35 43 51  8 16 24
## [24] 32 40 48  5 13 21 29 37 45  2 10 18 26 34 42 50  7 15 23 31 39 47  4
## [47] 12 20 28 36 44 52
```



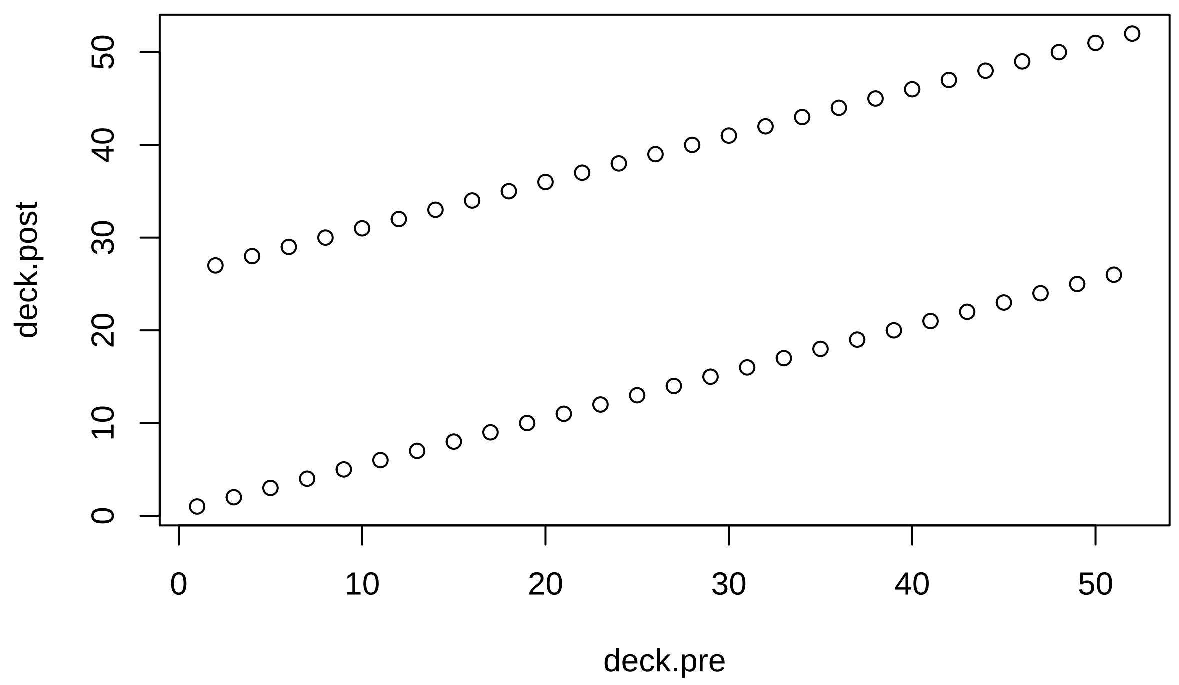
```
## [1] "Shuffles: 6"
##  [1]  1  5  9 13 17 21 25 29 33 37 41 45 49  2  6 10 14 18 22 26 30 34 38
## [24] 42 46 50  3  7 11 15 19 23 27 31 35 39 43 47 51  4  8 12 16 20 24 28
## [47] 32 36 40 44 48 52
```



```
## [1] "Shuffles: 7"
##  [1]  1  3  5  7  9 11 13 15 17 19 21 23 25 27 29 31 33 35 37 39 41 43 45
## [24] 47 49 51  2  4  6  8 10 12 14 16 18 20 22 24 26 28 30 32 34 36 38 40
## [47] 42 44 46 48 50 52
```



```
## [1] "Shuffles: 8"
##  [1]  1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18 19 20 21 22 23
## [24] 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46
## [47] 47 48 49 50 51 52
```



How Many Shuffles?

The number of perfect shuffles needed to return a deck to its original state is given by k where $2^k \equiv 1 \pmod{n-1}$. This is the deck's multiplicative order of 2 (mod $n-1$).

For the 52 card poker deck used above we have

```
n <- length(deck)

k <- 1
while (2^k %% (n-1) != 1){       ### Check for multiplicative order of 2 (mod n-1)
  print(paste(k, 2^k %% (n-1)))
  k <- k + 1
}
```

```
## [1] "1 2"
## [1] "2 4"
## [1] "3 8"
## [1] "4 16"
## [1] "5 32"
## [1] "6 13"
## [1] "7 26"
```

```
print(paste("To return the deck to its original state, use", k, "shuffles for", n, "cards."))
```

```
## [1] "To return the deck to its original state, use 8 shuffles for 52 cards."
```

For more information see: Weisstein, Eric W. "Out-Shuffle." From MathWorld–A Wolfram Web Resource.

http://mathworld.wolfram.com/Out-Shuffle.html